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Equations reducible to linear equations with constant coefficients

A Partial differential equation of the form

$$a_0 x^n \frac{\partial^n z}{\partial x^n} + a_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1} \cdot \partial y} + a_2 x^{n-2} y^2 \cdot \frac{\partial^n z}{\partial x^{n-2} \cdot \partial y^2} + \cdots$$

$$\cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

having variable co-efficients in a particular form as above (viz, the term $\frac{\partial^n z}{\partial x^n}$ is multiplied by the variable x^n. $\frac{\partial^n z}{\partial y^n}$ is multiplied by y^n, $\frac{\partial^n z}{\partial x^{n-r} \partial y^r}$ multiplied by $x^{n-r} \cdot y^r$ and so on)

Can be reduced to linear partial differential equation with constant coefficients by the substitution

$x = e^u$ and $y = e^v$ so that $u = \log x$ and $v = \log y$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{1}{x} \Rightarrow x \cdot \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}$$

$$\therefore x \cdot \frac{\partial}{\partial x} = \frac{\partial}{\partial u} = D$$

and $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial v} \cdot \frac{1}{y} \Rightarrow y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial v}$$

$$\therefore y \frac{\partial}{\partial y} = \frac{\partial}{\partial v} = D$$

Now $$x \cdot \frac{\partial}{\partial x}\left(x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = x\left[x^{n-1} \cdot \frac{\partial^n z}{\partial x^n} + \frac{\partial^{n-1} z}{\partial x^{n-1}} \cdot (n-1) \cdot x^{n-2}\right]$$

$$= x^n \cdot \frac{\partial^n z}{\partial x^n} + (n-1) \cdot x^{n-1} \cdot \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$\Rightarrow x^n \cdot \frac{\partial^n z}{\partial x^n} = \left(x \cdot \frac{\partial}{\partial x} - n + 1\right) \cdot x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$= (D - n + 1) \cdot x^{n-1} \cdot \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

Putting $n = 2, 3, 4 \ldots$ We have.

$$x^2 \frac{\partial^2 z}{\partial x^2} = (D-1)\, x \cdot \frac{\partial z}{\partial x} = (D-1) Dz = D(D-1)z$$

$$x^3 \frac{\partial^3 z}{\partial x^3} = (D-2)\, x^2 \frac{\partial^2 z}{\partial x^2} = (D-2)(D-1) Dz \text{ etc}$$

Similarly $\quad y^2 \dfrac{\partial^2 z}{\partial y^2} = D'(D'-1) z$. etc

and $\quad xy \dfrac{\partial^2 z}{\partial x \cdot \partial y} = DD'z$ etc.

Substituting in the given equation. it reduces to the $F(D, D')z = V$ which is a linear partial differential equation with constant coefficients and can be easily solved by the methods already discussed earlier.

Que ① – Solve : $x^2 \cdot \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = 0$

Sol'n :- Substituting $x = e^u$ and $y = e^v$ so that $u = \log x$ and $v = \log y$ and denoting $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ by D and D' respectively the given equation reduces to

$$[D(D-1) + 2DD' + D'(D'-1)] z = 0$$

$$[D^2 - D + 2DD' + D'^2 - D'] z = 0$$

$$\Rightarrow [\{D^2 + 2DD' + D'^2] - (D + D')] z = 0$$

$$\Rightarrow [(D+D)^2 - (D+D')] z = 0$$

$$\Rightarrow (D+D')(D+D'-1) z = 0.$$

Hence the Sol'n is

$$z = F_1(v-u) + e^u F_2(u+v)$$

$$= F_1(\log y - \log x) + x F_2(\log y + \log x)$$

$$= F_1(\log y/x)$$

$$= F_1\left(\log \frac{y}{x}\right) + x F_2(\log y/x)$$

$$= \phi_1(y/x) + x\, \phi_2(y/x)$$